

Exact null-controllability of interconnected abstract evolution systems by scalar force motion

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Abstract

The paper deals with exact null-controllability problem for a linear control system consisting of two serially connected abstract control systems. Controllability conditions are obtained. Applications to the exact null-controllability for interconnected control system of heat and wave equations are considered.

Keywords: Controllability, interconnected evolution equations, strongly minimal families.

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1 Introduction and problem statement

Many engineering applications generate interactive physical processes described by interconnected control systems. Control design for such systems modeled by interconnected partial differential control systems, have investigated intensively over the last years.

The goal of the present paper is to establish complete null controllability conditions for a control object containing two control abstract evolution systems interconnected into a series such that a control function from the second control system is an output of the first one.

Let X_1, X_2 be complex separable Hilbert spaces. Consider the control evolution equation [7],[11] with scalar control

$$\dot{x}_1(t) = A_1 x_1(t) + b_1 v(t), x_1(0) = x_1^0, \quad (1.1)$$

$$v(t) = (c, x_2(t)), 0 \leq t < +\infty, \quad (1.2)$$

where $x_2(t)$ is a mild solution of the another control equation of the form

$$\dot{x}_2(t) = A_2 x_2(t) + b_2 u(t), 0 \leq t < +\infty, x_2(0) = x_2^0. \quad (1.3)$$

Here $x_1(t), x_1^0, b_1 \in X_1$, where X_1 is the state space of equation (1.1), $v(t) \in \mathbb{C}, x_2(t), x_2^0, c, b_2 \in X_2$, where X_2 is the state space of equation (1.3), $u(t) \in \mathbb{C}$, and the linear operators A_1 and A_2 generate strongly continuous C_0 -semigroup $S_1(t)$ in X_1 and $S_2(t)$ in X_2 correspondingly [7, 11].

The formulas $b_1 v$ and $b_2 u, v, u \in \mathbb{C}$ express linear bounded operators from \mathbb{C} to X_1 and from \mathbb{C} to X_2 .

The interconnected system (1.1)–(1.3) is governed by a control $u(t)$ of equation (1.3).

2 Basic assumptions and definitions

1. The operator A_1 has purely point spectrums σ_1 with no finite limit points.

Since we use scalar controls we assume the geometrical multiplicity of eigenvalues of the operator A_1 to be equal to 1.

2. All eigenvectors of the operators A_1 produce a Riesz basic in their linear span.

Let the eigenvalues $\lambda_j \in \sigma_1, j = 1, 2, \dots$, be enumerated in the order of non-decreasing absolute values, let α_j be the algebraic multiplicities¹ of $\lambda_j \in \sigma_1$ correspondingly, and let $\varphi_{jk}, j \in \mathbb{N}, k = 1, 2, \dots, \alpha_j$ be the generalized eigenvectors of the operator $A_1, A_1 \varphi_{j\alpha_j} = \lambda_j \varphi_{j\alpha_j}, j \in \mathbb{N}$, and let $\psi_{jk}, j \in \mathbb{N}, k = 1, 2, \dots, \alpha_j$, be the generalized eigenvectors of the adjoint operator $A_1^*, A_1^* \psi_{j\alpha_j} = \bar{\lambda}_j \psi_{j\alpha_j}, j \in \mathbb{N}$, chosen such that

$$\begin{aligned} (\varphi_{s\alpha_s-l+1}, \psi_{jk}) &= \delta_{sj} \delta_{lk}, \\ s, j &\in \mathbb{N}, \quad l = 1, \dots, \alpha_s, \quad k = 1, \dots, \alpha_j. \end{aligned} \quad (2.1)$$

We use the following notations²:

$$\begin{aligned} x_{jk}(t) &= (x(t, x_0, u(\cdot)), \psi_{jk}), \quad x_{jk}^0 = (x_1^0, \psi_{jk}), \\ b_{1jk} &= (b_1, \psi_{jk}), \quad j \in \mathbb{N}, k = 1, 2, \dots, \alpha_j, \end{aligned} \quad (2.2)$$

$$\begin{aligned} g_{jk}(-t) &= e^{-\lambda_{1j}t} \sum_{l=0}^{\alpha_j-k} b_{1jk+l} \frac{(-t)^l}{l!}, \quad t \in [0, t_1], \\ j &\in \mathbb{N}, \quad k = 1, 2, \dots, \alpha_j. \end{aligned} \quad (2.3)$$

¹The geometric multiplicity is the number of Jordan blocks corresponding to $\lambda_j \in \sigma_1$. Throughout in the paper it is equal to 1.

²If $0 \in \sigma_1$, we denote $\lambda_0 = 0$ and in (2.1)–(2.7) $j \in 0 \cup \mathbb{N}$.

The following properties of sequences $\{x_j \in X_1, j = 1, 2, \dots\}$ are very significant throughout in the given paper.

Definition 2.1 *The sequence $\{x_j \in X_1, j = 1, 2, \dots\}$ is said to be minimal, if there no element of the sequence belonging to the closure of the linear span of others. By other words,*

$$x_j \notin \overline{\text{span}} \{x_k \in X_1, k = 1, 2, \dots, k \neq j\}.$$

Definition 2.2 *The sequence $\{x_j \in X_1, j = 1, 2, \dots\}$ is said to be strongly minimal, if there exists a positive number $\gamma > 0$ such that*

$$\gamma \sum_{k=1}^n |c_k|^2 \leq \left\| \sum_{k=1}^n c_k x_k \right\|^2, \quad n = 1, 2, \dots, \quad (2.4)$$

where

$$\gamma = \lim_{n \rightarrow \infty} \min_{\substack{c_1, \dots, c_n: \\ \sum_{k=1}^n |c_k|^2 = 1}} \left\| \sum_{k=1}^n c_k x_k \right\|^2.$$

Using above properties the following results have been proven in [13].

Definition 2.3 *Equation (1.1) is said to be exact null-controllable on $[0, t_1]$ by square integrable controls, if for each $x_{10} \in \mathfrak{X}_1$ and $\alpha \in \mathbb{R}$ there exists a control $u(\cdot) \in L_2[0, t_1]$, such that*

$$x_1(t_1, x_{10}, v(\cdot)) = 0. \quad (2.5)$$

Theorem 2.1 [13] *Let the sequence of eigenvectors of operator A_1 forms a Riesz basic in X_1 . Equation (1.1) is exact null-controllable on $[0, t_1]$ by scalar controls, if and only if the sequence (2.3) of generalized exponents is strongly minimal in $L_2([0, t_1])$.*

For the simplicity of the exposition we assume below that all the eigenvalues of the operator A_1 are simple. In this case the eigenvector of the operator A_1^* , corresponding to the eigenvalue λ_j , can be denoted by ψ_j , $j = 1, 2, \dots$, $b_{1j} = (b_1, \psi_j)$, $j = 1, 2, \dots$ and the family of generalized exponents (2.3) can be simplified and written by exponents

$$\left\{ g_j(-t) = b_{1j} e^{-\lambda_{1j} t}, j = 1, 2, \dots \right\}. \quad (2.6)$$

If $0 \in \sigma_1$, then according to our assumption 0 is a simple eigenvalue, and $\sigma_1 = \{\lambda_j, j = 0, 1, 2, \dots\}$, where $\lambda_0 = 0$. Otherwise $\sigma_1 = \{\lambda_j, j = 1, 2, \dots\}$. In both cases $\lambda_j \neq 0, j = 1, 2, \dots$, and

$$\left\{ g_j(-t) = b_{1j} e^{-\lambda_j t}, j = 0, 1, 2, \dots \right\} \quad (2.7)$$

3 Controllability of interconnected equations by force motions

The problem can be investigated by two different ways.

The first way is :

we construct a composite evolution equation

$$\begin{aligned} \dot{x} &= \mathbf{A}x(t) + \mathbf{b}u(t), \\ x(0) &= x_0, \end{aligned} \quad (3.1)$$

where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in X = X_1 \times X_2$, $x(0) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$,

$\mathbf{A} = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, where the linear operator $B : X_2 \rightarrow X_1$ is defined by

$$Bx_2 = b_1(c, x_2), \mathbf{b} = \begin{pmatrix} 0 \\ b_2 \end{pmatrix} \in X.$$

System (3.1) is considered as a system combining the features of both equations (1.1) and (1.3).

Next need to prove that system (3.1) is an equation of the form (1.1), i.e. the operator \mathbf{A} generates C_0 -semigroup and satisfies the conditions imposed on the operator A_1 (see page 2), and afterward one can use known controllability conditions of equation (1.1)

As a rule this way has been used in the literature for the case when both equations (1.1) and (1.3) are control PDE's.

Since the linear operator $Bx_2 = b_1(c, x_2)$ is obviously bounded one can prove that the operator \mathbf{A} generates C_0 -semigroup.

In order to continue we need to prove that assumptions on page 2 hold. It may be done, if the operator A_2 satisfies the same conditions on page 2. However in this paper we know nothing about the operator A_2 except that the operator A_2 generates a C_0 -semigroup.

It gives the motivation to use the different approach.

3.1 Controllability of equation (1.1) by smooth controls

Let $AC[0, t_1]$ be the space of absolutely continuous functions defined on the closed segment $[0, t_1]$, and let $a \in \mathbb{C}$. Denote:

$$\begin{aligned} H_\alpha^1[0, t_1] &= \{v(\cdot) \in AC[0, t_1], v(0) = a, v'(\cdot) \in L_2[0, t_1]\}, \\ H_{\alpha\beta}^1[0, t_1] &= \{v(\cdot) \in AC[0, t_1], v(0) = a, v(t_1) = \beta, v'(\cdot) \in L_2[0, t_1]\}, \\ H_0^1[0, t_1] &= H_{00}^1[0, t_1] \end{aligned}$$

We know nothing about differential properties of a generalized solution $x_2(t)$, generated by control $u(\cdot) \in L_2[0, t_1]$, but in accordance with the definition of a generalized solution of equation (1.3) the function $v(t) = (c, x_2(t))$ defined by (1.2) is absolutely continuous for any $c \in D(A_2^*)$ [1]. In order to keep the control object in the equilibrium state, we will turn off the control $v(t)$ at the end of the control process, i.e. $v(t) \equiv 0, t \geq t_1$. Hence to investigate the exact null-controllability of interconnected equations (1.1)-(1.3) it makes sense to consider the exact null controllability of equation (1.1) on $[0, t_1]$ by single smooth controls $v(\cdot)$ of the space $H_\alpha^1[0, t_1]$.

Definition 3.4 Equation (1.1) is said to be exact null-controllable on $[0, t_1]$ by smooth controls, if for each $x_{10} \in \mathfrak{X}_1$ and $\alpha \in \mathbb{C}$ there exists a control $v(\cdot) \in H_{\alpha 0}^1[0, t_1]$, such that

$$x_1(t_1, x_{10}, v(\cdot)) = 0. \quad (3.2)$$

To establish the controllability conditions by smooth controls, we need the following auxiliary result.

Lemma 3.1 The operator $\mathcal{A} = \begin{pmatrix} A_1 & b_1 \\ 0 & 0 \end{pmatrix}$ generates strongly continuous C_0 -semigroup in the product space $X_1 \times \mathbb{C}$.

Proof. Denote by $R(\mu)$ and $\mathcal{R}(\mu)$ the resolvent operators of the operators A and \mathcal{A} correspondingly.

Denote by $\mathcal{R}_0(\mu)$ the resolvent of the operator $\mathcal{A}_0 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$. Obviously

$$\left\| \mathcal{R}_0^n(\mu) \begin{pmatrix} x \\ v \end{pmatrix} \right\|^2 = \|R^n(\mu)x\|^2 + \left\| \frac{1}{\mu^n}v \right\|^2, \forall n \in \mathbb{N}.$$

In accordance with Hille-Iosida Theorem [7] there exist positive constants K, a , such that

$$\|R^n(\mu)\| \leq \frac{K_1}{(\mu - a)^n} \quad \forall \mu > a, \quad \forall n \in \mathbb{N},$$

Therefore

$$\begin{aligned} \left\| \mathcal{R}_0^n(\mu) \begin{pmatrix} x \\ v \end{pmatrix} \right\|^2 &\leq \frac{K^2}{(\mu - a)^{2n}} \|x\|^2 + \frac{1}{\mu^{2n}} \|v\|^2 \leq \\ &\leq \frac{K_1^2}{(\mu - a)^{2n}} (\|x\|^2 + \|v\|^2), \\ \forall \mu &> a, \forall n \in \mathbb{N}. \end{aligned}$$

where $K_1 = \max\{K, 1\}$. Hence

$$\|\mathcal{R}_0^n(\mu)\| \leq \frac{K_1}{(\mu - a)^n}, \quad \forall n \in \mathbb{N}, \forall \mu > a.$$

It shows [7] that the operator \mathcal{A}_0 generates C_0 -semigroups.

Denote $\mathcal{B} = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}$. Obviously the operator \mathcal{B} is bounded and $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$.

It is well-known [7], that if the operator \mathcal{A}_1 generates C_0 -semigroup and the operator \mathcal{B} is bounded, then the operator $\mathcal{A} = \mathcal{A}_1 + \mathcal{B}$ also generates C_0 -semigroup.

It proves the lemma.

Remark. Obviously, if equation (1.1) is exact null-controllable on $[0, t_1]$ by smooth controls $v(\cdot) \in H_{a0}^1[0, t_1]$, then, obviously, it is exact null-controllable on $[0, t_1]$ by $v(\cdot) \in L_2[0, t_1]$. According to Theorem 2.1 the family (2.7) of exponents should be strongly minimal. Surely it is impossible, if $0 \in \sigma_1$ and $b_{10} = 0$. Hence in the case of $0 \in \sigma_1$ it makes sense to consider only the condition $b_{10} \neq 0$.

Theorem 3.2 *Equation (1.1) is exact null-controllable on $[0, t_1]$ by smooth controls $v(\cdot) \in H_{a0}^1[0, t_1]$, if and only then either $0 \notin \sigma_1$ or $0 \in \sigma_1$ and $b_{10} \neq 0$, and family*

$$\left\{ 1, \frac{b_{1j}}{\lambda_j} e^{-\lambda_{1j}t}, j = 1, 2, \dots \right\} \quad (3.3)$$

of exponents is strongly minimal.

Proof. One can write system (1.1)) governed by smooth control $v(\cdot) \in H_{a0}^1[0, t_1]$, by

$$\dot{x}_1(t) = A_1 x_1(t) + b_1 v(t), x_1(0) = x_1^0, \quad (3.4)$$

$$\dot{v}(t) = u(t), v(0) = \alpha, \quad 0 \leq t < +\infty, \quad (3.5)$$

As far as the operator \mathcal{A} generates a strongly continuous C_0 -semigroup, composite system (3.9)–(3.5) can be written in the form of (1.1) as follows:

$$\dot{z}(t) = \mathcal{A}z(t) + \mathbf{b}v(t), \quad (3.6)$$

where $z = \begin{pmatrix} x \\ v \end{pmatrix} \in X_1 \times \mathbb{C}$, the operator \mathcal{A} is defined in Lemma 3.3,

$$\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $\sigma(\mathcal{A})$ be the spectrum of the operator \mathcal{A} . We have $\sigma(\mathcal{A}) = \sigma_1 \cup \{0\}$, $0 \notin \sigma$. One can see that the operator \mathcal{A} satisfies the assumption on page 2

Denote by $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \in X \times \mathbb{C}$ the eigenvector of the adjoint operator \mathcal{A}^* corresponding to the eigenvalue $\lambda \in \sigma(\mathcal{A})$. We have

$$\mathcal{A}^* = \begin{pmatrix} A_1^* & 0 \\ b_1^* & 0 \end{pmatrix},$$

where b_1^* is a linear functional from X_1 to \mathbb{C} , defined by

$$b_1^*x = (b_1, x), \forall x \in X_1.$$

The eigenvalues and corresponding eigenvectors of the operator \mathcal{A}^* are defined as follows:

$$(\bar{\lambda}I - \mathcal{A}^*)\psi = \begin{pmatrix} (\bar{\lambda}I - A_1^*)\psi^1 \\ -b^*\psi^1 + \bar{\lambda}\psi^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.7)$$

Equality (3.7) holds if and only if

$$(\bar{\lambda}I - \mathcal{A}^*)\psi^1 = 0, -b^*\psi^1 + \bar{\lambda}\psi^2 = 0. \quad (3.8)$$

If $\lambda \in \sigma_1$, and $\lambda \neq 0$, then from (3.7) it follows, that if $\psi^1 = 0$, then $\psi^2 = 0$ as well, but it is impossible, because ψ is an eigenvector. Hence ψ^1 is an eigenvector of A_1^* and in accordance with the theorem conditions and $\psi^2 = \frac{1}{\bar{\lambda}}(b_1, \psi^1)$. In this case the eigenvectors ψ_λ of the operator \mathcal{A}^* corresponding to its eigenvalue $\lambda \in \sigma_1$, $\lambda \neq 0$ are defined as follows:

$$\psi_\lambda = \begin{pmatrix} \psi^1 \\ \frac{1}{\bar{\lambda}}(b, \psi^1) \end{pmatrix},$$

where ψ^1 is an eigenvector of A_1^* , corresponding to an eigenvalue $\lambda \in \sigma$.

1. Let's continue to prove the theorem for the case $0 \notin \sigma_1$.

If $\lambda = 0$, then 0 is a regular value for A_1 , so from (3.8) it follows, that $\psi^1 = 0$, and therefore ψ^2 may be any nonzero constant. One can set $\psi^2 = 1$. Therefore in this case the eigenvalues and corresponding eigenvectors of the operator \mathcal{A}^* are defined as follows:

$$\psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Summary: let $\lambda_j, j = 0, 1, 2, \dots \in \sigma_1 \cup \{0\} = \sigma(\mathcal{A})$ enumerated by increasing of their absolute values.

The eigenvectors $\psi_j, j = 0, 1, 2, \dots$ of the operator \mathcal{A}^* are

$$\psi_j = \begin{pmatrix} \psi_j^1 \\ \frac{b_{1j}}{\lambda_j} \end{pmatrix}, j = 1, 2, \dots, \psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3.9)$$

where $\psi_j^1, j = 1, 2, \dots$, are eigenvectors of the operator A^* .

From (3.9) it follows, that

$$(\mathfrak{b}, \psi_j) = \begin{cases} 1, & j = 0, \\ \frac{b_{1j}}{\lambda_j}, & j = 1, 2, \dots \end{cases} \quad (3.10)$$

Hence one can see that the sequence (2.6) of generalized exponents for system (3.6) is exactly the sequence (3.3).

By Theorem 2.1 the exact null-controllability of system (3.6) holds true if and only if family (3.3) of exponents is strongly minimal.

The exact null-controllability of system (3.6) is completely equivalent to the exact null-controllability of equation (1.1) on $[0, t_1]$, $t_1 > 0$ by smooth controls.

It proves the theorem for the case $0 \notin \sigma_1$.

2. Prove the theorem for the case $0 \in \sigma_1$.

Let $\psi_0 = \begin{pmatrix} \psi_0^1 \\ \psi_0^2 \end{pmatrix}$ be an eigenvector of the operator \mathcal{A}^* , corresponding the eigenvalue $\lambda_0 = 0$. In accordance with (3.8) from (3.7) it follows, that

$$(-A_1^*) \psi_0^1 = 0, - (b_1, \psi_0^1) = 0.. \quad (3.11)$$

Let $\psi_0^1 \neq 0$. From (3.11) it follows, that ψ_0 is an eigenvector of the operator A_1^* , corresponding to the eigenvalue $\lambda_0 = 0$, and $b_{10} = (b_1, \psi_0^1) = 0, \psi_0^1 \neq 0$. This contradicts to theorem conditions.

Hence, $\psi_0^1 = 0$, so again corresponding eigenvector ψ_0 of the operator \mathcal{A}^* is defined as well as in the case $0 \notin \sigma_1$, i.e.

$$\psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Hence again (\mathfrak{b}, ψ_j) , $j = 1, 2, \dots$ is defined by (3.10)

The proof is finished as well as in the case $0 \notin \sigma_1$. ■

3.2 Controllability criterion of interconnected equations by force motion

Definition 3.5 *Interconnected system (1.1)–(1.3) is said to be exact null-controllable on $[0, t_1]$ if for each $x_1^0 \in X_1$, $x_{20} \in X_2$ there exists a control $u(\cdot) \in L_2[0, t_1]$, such that a mild solution $x_1(t, x_1^0, v(\cdot))$ of equation (1.1) with a control $v(t)$ defined by (1.2) satisfies the condition*

$$x_1(t, x_1^0, v(\cdot)) = 0. \quad (3.12)$$

3.2.1 Regular case $(c, b_2) \neq 0$

Theorem 3.3 *If*

- *the family of exponents (3.3) is strongly minimal,*
- *$c \in D(A_2^*)$ and $(c, b_2) \neq 0$,*

then interconnected equation (1.1)–(1.3) is exact null-controllable on $[0, t_1]$.

Proof. Let $c \in D(A_2^*)$ and $(c, b_2) \neq 0$. As much as family (3.3) of exponents is strongly minimal, then, according to Theorem 3.2, for any $x_{10} \in X_1$ and $\alpha \in \mathbb{C}$ there exists a control $v(\cdot) \in H_{\alpha 0}^1[0, t_1]$ such that (3.12) holds, and vice versa. Hence if any function $v(\cdot) \in H_{\alpha 0}[0, t_1]$ can be expressed by

$$v(t) - v(0) = (c, x_2(t)), t \in [0, t_1], \quad (3.13)$$

where

$$x_2(t) = S_2(t)x_{20} + \int_0^t S_2(t - \tau) b_2 u(\tau) d\tau, \quad (3.14)$$

then interconnected system (1.1)–(1.3) is exact null-controllable on $[0, t_1]$.

Obviously any function $v(\cdot) \in H_{a0}^1[0, t_1]$ can be expressed by (3.13) if and only if the Volterra integral equation of the first kind with continuous kernel $K(t - \tau) = (c, S_2(t - \tau) b_2)$

$$w(t) = \int_0^t (c, S_2(t - \tau) b_2) u(\tau) d\tau, t \in [0, t_1], \quad (3.15)$$

where $w(t) = v(t) - v(0)$, $v(0) = (c, x_{20})$, has a solution $u(\cdot) \in L_2[0, t_1]$ for any $w(\cdot) \in H_0^1[0, t_1]$.

One can use some classical conditions for the existence of solutions for equation (3.15) [14]. One of them are:

- 1) $w(t)$ is continuously differentiable and $w(0) = 0$,
- 2) the kernel $K(t - \tau) = (c, S_2(t - \tau) b_2)$ of equation (3.15) is continuously differentiable and $K(t, t) = (c, b_2) \neq 0$.

Since $c \in D(A_2^*)$, we have $\frac{\partial}{\partial \tau} K(t - \tau) = -(S_2^*(t - \tau) A_2^* c, b_2)$ to be continuous [7, 11], hence if absolutely continuous function $v(t)$ appears to be continuously differentiable, there exists a continuous solution of equation (3.15)[14].

If absolutely continuous function $v(t)$ is not continuously differentiable, we consider the integral Volterra equation of the second kind

$$w(t) = v(t) - v(0) = (c, b_2) U(t) + \int_0^t (S_2^*(t - \tau) A_2^* c, b_2) U(\tau) d\tau. \quad (3.16)$$

Because of $c \in D(A_2^*)$ the kernel $K_1(t - \tau) = (S_2^*(t - \tau) A_2^* c, b_2)$ is continuous [7, 11], hence equation (3.16) has a continuous solution $U(t)$ for any continuous function $v(t)$ [14]. This solution is obtained by [14]

$$U(t) = \frac{w(t)}{(A_2^* c, b_2)} + \int_0^t R(t - \tau) v(\tau) d\tau, \quad (3.17)$$

where $R(t)$ is the resolvent of equation (3.16), obtained by [14]

$$R(t - \tau) = \sum_{n=0}^{\infty} K_{n+1}(t - \tau), \quad 0 \leq \tau \leq t \leq t_1, \quad (3.18)$$

where $K_{n+1}(t - \tau)$ are repeated kernels defined by the recurrence

$$K_{n+1}(t - \tau) = \int_{\tau}^t K_1(t - \theta) K_n(\theta - \tau) d\theta, \quad n = 1, 2, \dots \quad (3.19)$$

and the series (3.17) converges uniformly. Hence $R(t - \tau)$ is continuous, so from (3.17) it follows, that the function $U(t)$ is absolutely continuous

function, i.e. there exists an integrable function $u(t)$, such that $u(t) = \dot{U}(t)$ a.e. for $t \in [0, t_1]$, and $U(0) = 0$. Actually because of square integrability of $\dot{v}(t)$ the function $u(t)$ appears to be square integrable. Using the integrating by parts we obtain by (3.12) and taking into account the condition $U(0) = 0$

$$\begin{aligned} (c, b_2) U(t) + \int_0^t (S_2^*(t-\tau) A_2^* c, b_2) U(\tau) d\tau &= \\ &= \int_0^t (c, S_2(t-\tau) b_2) u(\tau) d\tau = w(t), \end{aligned} \quad (3.20)$$

i.e. (3.15) holds for a function $u(t) \in L_2[0, t_1]$.

This proves the theorem. ■

Remark 1 A square integrable control $u(t)$ is a square integrable first derivative of an absolutely continuous solution $U(t)$ of the integral Volterra equation (3.16) of the second kind, where $v(t) \in H_{\alpha 0}^1[0, t_1]$ is a control satisfying condition (3.2) (see Definition 3.4).

3.2.2 Singular case $(c, b_2) = 0$

If $(c, b_2) = 0$, then above consideration are not applicable, because equation (3.16) appears to be an Volterra equation

$$w(t) = \int_0^t (S_2^*(t-\tau) A_2^* c, b_2) U(\tau) d\tau.$$

of the first kind with continuous kernel $K_1(t-\tau) = (S_2^*(t-\tau) A_2^* c, b_2)$. However if $c \in D(A_2^{*2})$ and $(A_2^* c, b_2) \neq 0$, then the proof of Theorem can be used, if everywhere in the proof to replace the vector $c \in D(A_2^*)$ by the vector $A_2^* c \in D(A_2^*)$. By this way the following results can be obtained.

Definition 3.6 Interconnected system (1.1)–(1.3) is said to be exact null-controllable on $[0, t_1]$ by distributions, if for each $x_1^0 \in X_1$, $x_{20} \in X_2$ there exists a distribution (generalized control) $u(\cdot)$, such that a mild solution $x_1(t, x_1^0, v(\cdot))$ of equation (1.1) with a control $v(t)$ defined by (1.2) satisfies the condition

$$x_1(t, x_1^0, v(\cdot)) = 0.$$

Theorem 3.4 *If*

- *the family of exponents*

$$\left\{ 1, \frac{b_{1j}}{\lambda_j} e^{-\lambda_{1j} t}, j = 1, 2, \dots \right\}$$

is strongly minimal,

- $c \in D(A_2^{*2}), (c, b_2) = 0$ and $(A_2^* c, b_2) \neq 0$,

then interconnected equation (1.1)–(1.3) is exact null-controllable on $[0, t_1]$ by distributions.

Proof. Let $c \in D(A_2^{*2})$ and $(A_2^* c, b_2) \neq 0$. Arguing as in the regular case, consider the Volterra equation

$$w(t) = v(t) - v(0) = (A_2 c, b_2) U(t) + \int_0^t (S_2^*(t - \tau) A_2^{*2} c, b_2) U(\tau) d\tau. \quad (3.21)$$

Again because of $(A_2^* c, b_2) \neq 0$ and $c \in D(A_2^{*2})$ the kernel $K_2(t, \tau) = (S_2^*(t - \tau) A_2^{*2} c, b_2)$ is continuous [7, 11], hence equation (3.21) has a continuous solution $U_1(t)$ for any continuous function $v(t)$ [14].

Since the function $w(t)$ is absolutely continuous with square integrable derivative, from (3.21) it follows that the function $U_1(t)$ is absolutely continuous, so as well as in the regular case

$$\begin{aligned} w(t) &= (A_2^* c, b_2) U_1(t) + \int_0^t (S_2^*(t - \tau) A_2^{*2} c, b_2) U_1(\tau) d\tau = \\ &= \int_0^t (S_2^*(t - \tau) A_2^* c, b_2) U(\tau) d\tau, \end{aligned}$$

where $U_1(0) = 0, U(t) = \dot{U}_1(t)$ a.e. for $t \in [0, t_1]$, and $U(\cdot) \in L_2[0, t_1]$ because of $\dot{v}(\cdot) \in L_2[0, t_1]$. If to continue the integration by parts for

$$\int_0^t (S_2^*(t - \tau) A_2^* c, b_2) U(\tau) d\tau,$$

we can only obtain (3.15) for $u(t)$ which is understood as a distribution (the first distributional (generalized) derivative of the square integrable function $U(t)$ or the second distributional derivative of the continuous function $U_1(t)$).

This proves the theorem. ■

The same approach can be used, if $c \in D(A_2^{*m})$, $(A_2^{*k} c, b_2) = 0, k = 0, 1, \dots, m-1$, $(A_2^{*m-1} c, b_2) \neq 0$ for some $m \in \mathbb{N}$.

Remark. Using the Laplace Transform in (3.15) we obtain for sufficiently large $\text{Re } s$

$$W(s) = \left(c, (sI - A_2)^{-1} b_2 \right) U(s) \quad (3.22)$$

As much as $w(\cdot) \in H_{0t_1}^1[0, t_1]$ and $v(t) \equiv 0, t > t_1$, the Laplace Transform $W(s)$ of the control $w(t)$ exists. Clearly, for each $W(s)$ equation (3.22) is solvable with respect to $U(s)$ if and only if the scalar function $\left(c, (sI - A_2)^{-1} b_2 \right)$ does not identically equal to zero for sufficiently large $\text{Re } s$. It surely holds true in regular case. .

3.3 Dual controllability criterion

Now consider the case $c \notin D(A_2^*)$.

If $c \notin D(A_2^*)$ then it is impossible to provide the existence of continuous solution for equation (3.15).

Obviously

$$(x_2^*, S_2(t - \tau) x_2) = (S_2^*(t - \tau) x_2^*, x_2), \quad \forall x_2, x_2^* \in X_2. \quad (3.23)$$

Hence if $b_2 \in D(A_2)$, then the function $(c, S_2(t - \tau) b_2)$ is continuously differentiable and

$$\frac{d}{dt} (c, S_2(t - \tau) b_2) = (c, S_2(t - \tau) A_2 b_2), \quad \tau \in [0, t]. \quad (3.24)$$

Using (3.23) in (3.15) we obtain that

$$w(t) = \int_0^t (c, S_2(t - \tau) b_2) u(\tau) d\tau \quad (3.25)$$

If $b \in D(A_2)$, then $\frac{\partial}{\partial \tau} K(t - \tau) = -(c, S_2(t - \tau) A_2 b_2)$ is continuous [7, 11], hence if absolutely continuous function $v(t)$ appears to be continuously differentiable, there exists a continuous solution of equation (3.25)

If absolutely continuous function $v(t)$ is not continuously differentiable, we consider the integral Volterra equation of the second kind

$$\frac{w(t)}{(c, b_2)} = U(t) + \int_0^t \frac{(c, S_2(t - \tau) A_2 b_2)}{(c, b_2)} U(\tau) d\tau. \quad (3.26)$$

Arguing as well as in the proof of the Theorem 3.3 we obtain the validity of the following theorem:

Theorem 3.5 *If*

- *the family of exponents*

$$\left\{ 1, \frac{b_{1j}}{\lambda_j} e^{-\lambda_{1j} t}, j = 1, 2, \dots \right\} \quad (3.27)$$

is strongly minimal,

- $b_2 \in D(A_2)$ and $(c, b_2) \neq 0$,

then interconnected equation (1.1)–(1.3) is exact null-controllable on $[0, t_1]$, where the control $u(t)$ satisfying (3.6) is a square integrable first derivative of the absolutely continuous solution $U(t)$ of the integral Volterra equation of the second kind

$$w(t) = (c, b_2) U(t) + \int_0^t (c, S_2(t - \tau) A_2 b_2) U(\tau) d\tau, \quad (3.28)$$

The generalization for the case $(c, b_2) = 0$ is done as above (see the previous subsection).

3.4 Strong minimality of real exponential families.

A direct proof of this fact for a given sequence of exponents can sometimes be tough. Below we prove two lemmas which substantially facilitate the establishment of the strong minimality for real exponential families.

Lemma 3.2 *If $\mu_n > 0$, the series*

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n} \quad (3.29)$$

converges and the Dirichle series

$$\sum_{n=1}^{\infty} \frac{e^{-\mu_n \alpha}}{\beta_n} \quad (3.30)$$

converges for some $\alpha > 0$, then the sequence

$$\{\beta_n e^{\mu_n t}, n = 1, 2, \dots, t \in [0, t_1], \forall t_1 > 0\} \quad (3.31)$$

is strongly minimal.

Proof. Let $t_1 = 2t_2$. Using results of [5] one can show that if the series $\sum_{n=1}^{\infty} \frac{1}{\mu_n}$ converges and $\lambda_{n+1} - \lambda_n \geq 1$, then the sequence

$$\{e^{-\mu_n t}, n = 1, 2, \dots, t \in [0, t_2], \forall t_2 > 0\} \quad (3.32)$$

is minimal. Clearly the sequence

$$\{\beta_n e^{\mu_n t}, n = 1, 2, \dots, t \in [0, t_2], \forall t_2 > 0\}$$

is also minimal. In virtue of Theorem 1.5 of [5] for each $\varepsilon > 0$ there exists a positive constant K_ε such that the sequence $\{w_n(t), n = 1, 2, \dots, t \in [0, t_2]\}$ biorthogonal to the sequence (3.31) satisfies the condition

$$\|w_n(\cdot)\| < K_\varepsilon e^{\varepsilon \mu_n}, n = 1, 2, \dots,$$

Hence the sequence $\left\{u_n(t) = \frac{1}{\beta_n} w_n(t_2 - t) e^{-\mu_n t_2}, n = 1, 2, \dots, t \in [0, t_2]\right\}$ is biorthogonal to the sequence (3.26) and it satisfies the condition

$$\|u_n(\cdot)\| < \frac{1}{\beta_n} K_\varepsilon e^{\varepsilon \mu_n} e^{-\mu_n t_2} < \frac{1}{\beta_n} K_\varepsilon e^{\varepsilon \mu_n}, n = 1, 2, \dots, \quad (3.33)$$

The positive constant ε can be chosen such that $t_2 - \varepsilon > \alpha$.

By the Minkowsky inequality and (3.33) one can show that

$$\begin{aligned} & \sum_{n=1}^p \sum_{m=1}^p c_n e^{-\mu_n t_2} \left(\int_0^{t_2} u_n(t) u_m(t) dt \right) e^{-\mu_m t_2} c_m = \\ &= \int_0^{t_2} \left(\sum_{n=1}^p c_n e^{-\mu_n t_2} u_n(t) \right)^2 dt \leq \int_0^{t_2} \sum_{n=1}^p |c_n|^2 \sum_{n=1}^p |e^{-\mu_n t_2} u_n(t)|^2 dt = \\ &= \sum_{n=1}^p |c_n|^2 \sum_{n=1}^p \int_0^{t_2} |e^{-\mu_n t_2} u_n(t)|^2 dt = \\ &= \sum_{n=1}^p |c_n|^2 \sum_{n=1}^p e^{-2\mu_n t_2} \int_0^{t_2} |u_n(t)|^2 dt \leq \\ &\leq \sum_{n=1}^p |c_n|^2 \sum_{n=1}^p e^{-2\mu_n t_2} \|u_n(\cdot)\|^2 \leq \sum_{n=1}^p |c_n|^2 K_\varepsilon^2 \sum_{n=1}^p \frac{1}{\beta_n^2} e^{-2\mu_n(t_2 - \varepsilon)}. \end{aligned}$$

It is well-known from the Dirichle series theory [12], that if the Dirichle series (3.30) converges, then the Dirichle series $\sum_{n=1}^{\infty} \frac{e^{-\mu_n t}}{\beta_n}$ converges for any $t \geq \alpha$. Therefore according to theorem condition the series $\sum_{n=1}^{\infty} \frac{1}{\beta_n} e^{-2\mu_n(t_2 - \varepsilon)}$ converges³ for any $t_2, \varepsilon, t_2 > \varepsilon + \alpha$, so the same holds true for the series

³The number $\alpha_0 = \inf \{\alpha \in \mathbb{R} : \text{the series (3.30) converges}\}$ is said to be the abscissa of the convergence of Dirichle series [12].

$\sum_{n=1}^{\infty} \frac{1}{\beta_n^2} e^{-2\mu_n(t_2-\varepsilon)}$. Hence $\sum_{n=1}^p \frac{1}{\beta_n^2} e^{-2\mu_n(t_2-\varepsilon)} \leq M$, where M is a positive constant, so

$$\sum_{n=1}^p \sum_{m=1}^p c_n e^{-\mu_n t_2} \left(\int_0^{t_2} u_n(t) u_m(t) dt \right) e^{-\mu_m t_2} c_m \leq K_\varepsilon^2 M \sum_{n=1}^p |c_n|^2 \quad (3.34)$$

for every finite sequence $\{c_1, c_2, \dots, c_p\}$.

The sequence

$$\{h_n(t), n = 1, 2, \dots, t \in [0, t_1],$$

where⁴

$$h_n(t) = \begin{cases} e^{-\mu_n t_2} u_n(t - t_2), & t \in [t_2, 2t_2], \\ 0, & t \in [0, t_2], \end{cases} \quad n = 1, 2, \dots, \quad (3.35)$$

is the biorthogonal to the sequence

$$\{\beta_n e^{\mu_n t}, n = 1, 2, \dots, t \in [0, t_1]\}.$$

Indeed,

$$\begin{aligned} \int_0^{t_1} \beta_n e^{\mu_n t} h_m(t) dt &= \int_{t_2}^{2t_2} \beta_n e^{\mu_n t} e^{-\mu_m t_2} u_m(t - t_2) dt = \\ &= e^{(\mu_n - \mu_m)t_2} \int_0^{t_2} \beta_n e^{\mu_n \tau} u_m(\tau) d\tau = \delta_{nm}, n, m = 1, 2, \dots, \end{aligned}$$

where $\delta_{nm}, n, m = 1, 2, \dots$, is the Kroneker Delta.

Further we have

$$\begin{aligned} \int_0^{t_1} h_n(t) h_m(t) dt &= e^{-\mu_n t_2} \left(\int_{t_2}^{2t_2} u_n(t - t_2) u_m(t - t_2) dt \right) e^{-\mu_m t_2} = \\ &= e^{-\mu_n t_2} \left(\int_0^{t_2} u_n(t) u_m(t) dt \right) e^{-\mu_m t_2}, \end{aligned} \quad (3.36)$$

and so it follows from (3.34)–(3.36), that

$$\sum_{n=1}^p \sum_{m=1}^p c_n \left(\int_0^{t_1} h_n(t) h_m(t) dt \right) c_m \leq K_\varepsilon^2 M \sum_{n=1}^p |c_n|^2.$$

Hence [9]

$$\sum_{n=1}^p \sum_{m=1}^p c_n \left(\int_0^{t_1} \beta_n e^{\mu_n t} \beta_m e^{\mu_m t} \right) c_m d\tau \geq \gamma \sum_{n=1}^p |c_n|^2, p = 1, 2, \dots,$$

⁴Recall, that $t_1 = 2t_2$.

for every finite sequence $\{c_1, c_2, \dots, c_p\}$, where $\gamma = \frac{1}{K_\varepsilon^2 M} > 0$. It proves that the sequence

$$\{\beta_n e^{\mu_n t}, t \in [0, t_1], n = 1, 2, \dots\}$$

is strongly minimal for any $t_1 > 0$. QED ■

Lemma 3.3 *If conditions of Lemma 3.1 hold, then the sequence*

$$\{1, \beta_n e^{\mu_n t}, n = 1, 2, \dots, t \in [0, t_1], \forall t_1 > 0\} \quad (3.37)$$

is also strongly minimal.

Proof. One can write family (3.37) by

$$g_n e^{v_n t}, n = 0, 1, 2, \dots, \quad (3.38)$$

where

$$\begin{aligned} v_n &= \begin{cases} \alpha & n = 0 \\ \mu_n + \alpha & n = 1, 2, \dots \end{cases}, \\ g_n &= \begin{cases} 1, & n = 0 \\ \beta_n & n = 1, 2, \dots \end{cases}. \end{aligned}$$

One can see that the family (3.38) is the family of the form (3.31), and from the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\beta_n} e^{-\mu_n(t_2 - \varepsilon)}$ it follows that the series

$\sum_{n=1}^p \frac{1}{g_n^2} e^{-2(\mu_n + \alpha)(t_1 - \varepsilon)}$ also converges for any $t_2, \varepsilon, t_1 > \varepsilon$. Hence in accordance with Lemma 3.2 the sequence

$$\left\{ g_n e^{v_n t} = \begin{cases} e^{\alpha t}, & n = 0, \\ g_n e^{(\mu_n + \alpha)t}, & n = 1, 2, \dots, \end{cases}, t \in [0, t_2] \right\} \quad (3.39)$$

is strongly minimal for any $t_1 > 0$, i.e. there exists a constant $\gamma > 0$ such that

$$\left\| \sum_{n=0}^p c_n g_n e^{v_n t} \right\|^2 \geq \gamma \left\| \sum_{n=0}^p c_n \right\|^2. \quad (3.40)$$

As much as $\alpha > 0$, we obtain by (3.39)–(3.40), that

$$\begin{aligned} \left\| (c_0 + \sum_{n=1}^p c_n g_n e^{\mu_n t}) \right\|^2 &= \left\| e^{-\alpha t} \sum_{n=0}^p c_n g_n e^{v_n t} \right\|^2 \geq \\ &\geq e^{-2\alpha t_1} \left\| (c_0 + \sum_{n=1}^p c_n g_n e^{\mu_n t}) \right\|^2 \geq \gamma_\alpha \left\| \sum_{n=0}^p c_n \right\|^2, \forall p = 1, 2, \dots, \end{aligned}$$

where $\gamma_\alpha = \gamma e^{-2\alpha t_1} > 0$.

It proves the Lemma. ■

4 Examples. Exact null controllability of interconnected Heat Equation and Wave Equation by force motion

We consider the heat equation with force motion

$$y'_t = y''_{xx} + b_1(x) v(t), \quad 0 \leq t \leq t_1, \quad 0 \leq x \leq \pi, \quad (4.1)$$

$$y(0, t) = y(\pi, t) = 0, \quad 0 \leq t \leq t_1, \quad (4.2)$$

$$y(x, 0) = \varphi_0(x), \quad 0 \leq x \leq \pi, \quad (4.3)$$

governed by a control $u(\cdot)$ of the wave equation

$$z''_{tt} - z''_{xx} = b_2(x) u(t), \quad 0 \leq t \leq t_1, \quad 0 \leq x \leq \pi, \quad (4.4)$$

$$z(0, t) = z(\pi, t) = 0, \quad 0 \leq t \leq t_1, \quad (4.5)$$

$$z(x, 0) = \psi_0(x), \quad z'_t(x, 0) = \psi_1(x), \quad 0 \leq x \leq \pi, \quad (4.6)$$

Here $\varphi_0, \psi^j, j = 1, 2$, and $b_1(x), b_2(x)$ belong to $L_2[0, \pi]$.

Let $H^2[0, \pi]$, $H_0^1[0, \pi]$ be Sobolev spaces (see [8] for definitions of the spaces $H^m[a, b]$, $H_0^m[a, b]$, $a, b \in \mathbb{R}$.)

Heat equation (4.1)–(4.3) can be written in the semigroup framework (1.3), where $X_1 = L_2[0, \pi]$, the operator A_1 is defined by the differential operator $A_1 = \frac{d^2}{dx^2}$ with the domain ⁵

$$D(A_1) = H^2[0, \pi] \cap H_0^1[0, \pi]. \quad (4.7)$$

Denote: $z_1(x, t) = z(x, t)$, $z_2(x, t) = z'_t(x, t)$,

$$X_2 = H_0^1[0, \pi] \times L_2[0, \pi] = \{(z_1, z_2) : z_1 \in H_0^1[0, \pi], z_2 \in L_2[0, \pi]\}$$

with the scalar product of $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in X_2$ defined by

$$\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = \int_0^\pi (z'_1(x) y'_1(x) + z_2(x) y_2(x)) dx.$$

⁵ $D(A_1)$ can also be defined by [1]

$$D(A_1) = \left\{ \begin{array}{l} y(\cdot), y'(\cdot) \in AC[0, \pi], \\ y''(\cdot) \in L_2[0, \pi], y(0) = y(\pi) = 0, \end{array} \right\}$$

where $AC[0, \pi]$ is the set of absolutely continuous on $[0, \pi]$ functions.

According to [8], $H_0^1[0, \pi] = \{y(\cdot) \in AC[0, \pi], y(0) = y(\pi) = 0\}$.

Wave equation (4.4)–(4.6) can be written in the semigroup framework (1.3), where $x_2(t) = \begin{pmatrix} z_1(x, t) \\ z_2(x, t) \end{pmatrix} = \begin{pmatrix} z(x, t) \\ z'_t(x, t) \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ b_2(x) \end{pmatrix}$.

The operator A_2 is defined by the matrix differential operator $A_2 = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}$ with the domain

$$D(A_2) = (z_1(x), z_2(x)) \in (H^2[0, \pi] \cap H_0^1[0, \pi]) \times H_0^1[0, \pi], \quad (4.8)$$

$$A_2 z = A_2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_1'' \end{pmatrix}, \quad (4.9)$$

A smooth distributed control $v(t)$ in the force motion term of heat equation (4.1)–(4.3) can be considered as an observation

$$v(t) = \int_0^\pi (c'_1(x) z'_x(x, t) + c_2(x) z'_t(x, t)) dx, \quad 0 \leq t \leq t_1\pi, c_2(\cdot) \in L_2[0, \pi] \quad (4.10)$$

of wave equation (4.4)–(4.6), where $c(\cdot) = (c_1(\cdot), c_2(\cdot)) \in H_0^1[0, \pi] \times L_2[0, \pi]$.

Therefore the results obtained in the previous section can be applied.

4.0.1 Controllability conditions

The eigenvalues λ_n and corresponding eigenvectors $\varphi_n, n = 1, 2, \dots$ of the operator A_1 are obtained by $\lambda_n = -n^2, \varphi_n = \sin nx, n = 1, 2, \dots$, and as much as the operator A_1 is selfadjoint, the eigenvectors φ_n and the eigenvectors $\psi_n, n = 1, 2, \dots$, of the adjoint operator A_1^* are the same.

Obviously, the sequence $\sin nx, n = 1, 2, \dots$ of eigenvectors of the operator A_1 (or A_1^*) forms a Riesz basic in X_1 .

Denote $b_{1n} = \int_0^\pi b_1(x) \sin nx, n = 1, 2, \dots$.

In accordance with Theorem 3.3 to establish the conditions of the exact null controllability of interconnected system under consideration we should prove that the family

$$\left\{ 1, \frac{b_{1n}}{n^2} e^{n^2 t}, n = 1, 2, \dots, t \in [0, t_1] \right\} \quad (4.11)$$

is strongly minimal.

In our case $\mu_n = \lambda_n = -n^2, n = 1, 2, \dots$, the series $\sum_{n=1}^\infty \frac{1}{\mu_n} = \sum_{n=1}^\infty \frac{1}{n^2}$ converges.

Hence in accordance with Theorems 3.2–3.3 and Lemma 3.1 we obtain the validity of the following theorem:

Theorem 4.6 *If the series*

$$\sum_{n=1}^{\infty} \frac{n^2}{b_{1n}} e^{-n^2 \alpha} \quad (4.12)$$

converges for some $\alpha > 0$, then equation (4.1)–(4.3) is exact null-controllable on $[0, t_1]$, $\forall t_1 > 0$, by smooth controls $v(\cdot) \in H_{\alpha 0}^1[0, t_1]$.

4.1 Regular case

Theorem 4.7 *If*

1. *series (4.12) converges for some $\alpha > 0$,*
2. *$b_2(\cdot), c_2(\cdot) \in L_2[0, \pi]$,*
3. *$\int_0^\pi c_2(x) b_2(x) dx \neq 0$,*

then interconnected system equation (4.1)–(4.3), (4.4)–(4.6), interconnected by (4.10) is exact null-controllable on $[0, t_1]$, $\forall t_1 > 0$.

Proof. We have here $c(\cdot) = \begin{pmatrix} c_1(\cdot) \\ c_2(\cdot) \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ b_2(\cdot) \end{pmatrix}$. Therefore $b_2 \in X_2$ is equivalent to $b_2(\cdot) \in L_2[0, \pi]$, and $(c, b_2) = \int_0^\pi c_2(x) b_2(x) dx$, $A_2 b_2 = A_2 \begin{pmatrix} 0 \\ b_2(\cdot) \end{pmatrix} = \begin{pmatrix} b_2(\cdot) \\ 0 \end{pmatrix}$, so $b_2 = \begin{pmatrix} 0 \\ b_2(x) \end{pmatrix} \in D(A_2)$ for any $b_2(\cdot) \in L_2[0, \pi]$ and $(c, b_2) \neq 0$. The theorem follows from Theorem 3.5. We have here the regular case.

Remark. The theorem assertion is valid for any function $c_1(\cdot) \in H_0^1[0, \pi]$. ■

For example, conditions of Theorem 4.7 hold true for $b_1(x) = b_2(x) = x$, $c_2(x) = 1$, $x \in [0, \pi]$. In this case $b_{1n} = \int_0^\pi x \sin nx dx = \frac{(-1)^{n+1} \pi}{n}$, the series $\sum_{n=1}^{\infty} \frac{n^2}{b_{1n}} e^{-n^2 \alpha} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{\pi} e^{-n^2 \alpha}$ converges for any $\alpha > 0$; $b_2(\cdot) \in L_2[0, \pi]$, and $\int_0^\pi c_2(x) b_2(x) dx = \int_0^\pi x dx = \frac{\pi^2}{2}$.

4.2 Singular case

Let $c_2(\cdot), b_2(\cdot) \in L_2[0, \pi]$, but $\int_0^\pi c_2(x) b_2(x) dx = 0$. In this case $(c, b_2) = 0$, so Theorems 3.3 or 3.5 are not applicable, and it is impossible to provide to exact null controllability of interconnected system being considered.

Theorem 4.8 *If*

1. *series (4.12) converges for some $\alpha > 0$,*
2. *Either $b_2(\cdot) \in H^2[0, \pi] \cap H_0^1[0, \pi]$ or $c_1(\cdot), c_2(\cdot) \in H^2[0, \pi] \cap H_0^1[0, \pi]$,*
3. *$\int_0^\pi c_1(x) b_2(x) dx \neq 0$,*

then system (4.1)–(4.3), (4.4)–(4.6), interconnected by (4.10) is exact null-controllable on $[0, t_1], \forall t_1 > 0$ by distributions.

Proof. We have here $A_2^* = \begin{pmatrix} 0 & \frac{d^2}{dx^2} \\ 1 & 0 \end{pmatrix}$, $A_2^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2'' \\ z_1 \end{pmatrix}$, $c(\cdot) = \begin{pmatrix} c_1(\cdot) \\ c_2(\cdot) \end{pmatrix}$, $b_2 = \begin{pmatrix} 0 \\ b_2(\cdot) \end{pmatrix}$. Therefore
if $b_2(\cdot) \in H^2[0, \pi] \cap H_0^1[0, \pi]$, then
 $A_2 b_2 = A_2 \begin{pmatrix} 0 \\ b_2(\cdot) \end{pmatrix} = \begin{pmatrix} b_2(\cdot) \\ 0 \end{pmatrix}$, $A_2^2 b_2 = \begin{pmatrix} 0 \\ b_2''(\cdot) \end{pmatrix}$, so
 $b_2 = \begin{pmatrix} 0 \\ b_2(\cdot) \end{pmatrix} \in D^2(A_2)$;
if $c_1(\cdot), c_2(\cdot) \in H^2[0, \pi] \cap H_0^1[0, \pi]$, then
 $A_2^* c_2 = A_2^* \begin{pmatrix} c_1(\cdot) \\ c_2(\cdot) \end{pmatrix} = \begin{pmatrix} c_2''(\cdot) \\ c_1(\cdot) \end{pmatrix}$, $A_2^{*2} c = A_2^* \begin{pmatrix} c_2''(\cdot) \\ c_1(\cdot) \end{pmatrix} = \begin{pmatrix} c_1''(\cdot) \\ c_2''(\cdot) \end{pmatrix}$,
so $c \in D^2(A_2^*)$.

In both cases we have $(c, b_2) = 0$, $(c, A_2 b_2) = \int_0^\pi c_1(x) b_2(x) dx$, so according to the third condition of the theorem $(c, A_2 b_2) \neq 0$. Hence the theorem follows from Theorems 3.4. We have here the singular case. ■

For example, conditions of Theorem 4.8 hold true for $b_1(x) = b_2(x) = x$, $c_1(x) = x(\pi - x)$, $c_2(x) = 0, x \in [0, \pi]$.

5 Conclusion

Exact null-controllability conditions for two interconnected abstract control equations (1.1)–(1.3) governed by a control $u(t)$ of equation (1.3) are obtained.

Of course, these results can be extended for series of a number interconnected equations, governed by a control of the last one.

The case of simply eigenvalues has been considered for the sake of simplicity only.

The main problems allowing to obtain controllability results of the paper are :

1. Establishing of the exact null-controllability of equation (1.1) by smooth control.
2. Solvability conditions of Volterra integral equation (3.25) of the first kind and of the convolution type.

Both these problems are independent on each other. The mutual independence of these problems allow us to use the abstract approach developed in the given paper for investigation of various control problems for interconnected systems contained equations of a different structure. For example, equation (1.1) may be a parabolic control equation, governed by force motion control, and equation (1.3) may be a linear differential control system with delays [2, 6], governed by force motion control, and so on⁶.

The singular case does not seem to be essential (in our opinion), because there are a lot of practical situations, for which equation (1.1)–(1.3) and (4.4)–(4.6) are given, and need to decide, how to connect them. It means, that if the case $(c, b_2) = 0$ occurs, one can always choose other vector c , slightly different from the first one, such that $(c, b_2) \neq 0$.

The exact null-controllability for interconnected heat–wave equations is considered as illustrative example only. Surely the controllability problems for many kinds of control PDE’s have been extensively investigated in last years.

In our private opinion, a great majority of them can be investigated by the abstract approach presented in the given paper.

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⁶As is known the author the existing results in this field are devoted to couple PDE’s.

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